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Two dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory on the lattice via dimensional reduction

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Abstract: The $\mathcal{N} = (2, 2)$ extended super Yang-Mills theory in 2 dimensions is formulated on the lattice as a dimensional reduction of a 4 dimensional lattice gauge theory. We use the plaquette action for a bosonic sector and the Wilson- or the overlap-Dirac operator for a fermion sector. The fermion determinant is real and, moreover, when the overlap-Dirac operator is used, semi-positive definite. The flat directions in the target theory become compact and present no subtlety for a numerical integration along these directions. Any exact supersymmetry does not exist in our lattice formulation; nevertheless we argue that one-loop calculable and finite mass counter terms ensure a supersymmetric continuum limit to all orders of perturbation theory.

Keywords: Renormalization Regularization and Renormalons, Extended Supersymmetry, Field Theories in Lower Dimensions, Lattice Gauge Field Theories.

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Contents

1. Introduction

In this paper, we consider a lattice formulation of the $\mathcal{N} = (2, 2)$ super Yang-Mills (SYM) theory in 2 dimensions,[1] which is one of subjects of recent developments [1−3] (see also refs. [4−8] for recent related works).[2] In these works, ingenious constructions, based on the orbifolding and deconstruction [1], topological field theoretical representations [2] and the twisted supersymmetry and a geometrical discretization [3], respectively, are applied to find lattice actions that are invariant under a nilpotent supersymmetry.[3] Then it is found that, at least in 2 dimensional extended SYM theories, invariance under a full set of supersymmetry is restored in the continuum limit without any tuning of parameters.[4]

However, if one considers a numerical implementation of those constructions in refs. [1−3], a fact that the fermion determinant in these formulations is not guaranteed to be real (see ref. [4], for example) may pose a serious problem. In this paper, from a quite different viewpoint, we propose yet another lattice formulation of the $\mathcal{N} = (2, 2)$ SYM theory in 2 dimensions which is free from this complex determinant problem. By doing this, we aim at a rather practical (if not theoretically intriguing) lattice formulation of this system.

[1] This theory is sometimes referred to as the $\mathcal{N} = 2$ SYM theory in 2 dimensions, though this usage of terminology is somewhat confusing.

[2] In the present paper, the gauge group G is taken to be $\mathrm{SU}(N_c)$.

[3] For an application of the twisted supersymmetry to lattice supersymmetric theories from a somewhat different viewpoint, see ref. [9].

[4] This general strategy for supersymmetric theories on the lattice was advocated in ref. [10]. For the preceding works with a similar strategy, see ref. [11]. Analyses in refs. [1, 2] show that 3 dimensional extended SYM theories generically require a tuning of parameters.

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Our basic idea was inspired by a work of Fujikawa [12] and proceeds as follows: The spacetime lattice provides an ultraviolet (UV) cutoff for correlation functions. In perturbation theory with this lattice regularization, an integrand of a Feynman integral associated to a Feynman diagram is modified by the lattice spacing a so that the integral is UV convergent. Now suppose that we have a Feynman diagram in the continuum theory whose associated Feynman integral is UV finite. For such an integral, we may remove the lattice cutoff as $a \to 0$ because the integral must be independent of any UV cutoff when it is sent to infinity. This argument indicates that, in the $a \to 0$ limit, only potentially UV diverging Feynman integrals and correlation functions are influenced by details of a lattice formulation.

Our present target theory, the $\mathcal{N} = (2, 2)$ SYM theory in 2 dimensions, is perturbatively super-renormalizable. According to the standard power counting, besides vacuum bubble diagrams, only one-loop one- and two-point functions of bosonic fields are potentially UV diverging. One-point functions (tadpoles) are forbidden by the gauge invariance which will be manifest in our lattice formulation. Hence, only one-loop two-point functions of bosonic fields, which are potentially logarithmically diverging, may be influenced by a lattice regularization. In a power series expansion of these two-point functions with respect to the external momentum, only the first constant term is logarithmically diverging and the rest are UV finite. The gauge invariance again forbids this first term for gauge fields. We thus expect that only mass terms of scalar fields are influenced by a lattice formulation. In other words, in the $a \to 0$ limit, any trail of a lattice formulation, in particular a breaking of the supersymmetry in our present problem, will be eliminated by tuning a coefficient of scalar mass terms. Moreover, this tuning will be calculable in *one-loop* perturbation theory.[5]

Lattice artifacts are of $O(a)$ and scalar two-point functions diverge as $O(\log a)$ at most. Thus we expect that any trail of a lattice formulation in the $a \to 0$ limit, even if it exists, is of $O(a^0)$. In summary, we expect that an addition of mass counter terms of scalar fields, whose coefficient is calculable in the one-loop order and UV finite, ensures a supersymmetric continuum limit. No further tuning of parameters will be required.

It turns out that the above argument based on the continuum power counting is a little bit too naive. If one makes an "exotic" choice of a lattice action, there can appear certain diagrams, being peculiar to lattice perturbation theory, that are not covered by the continuum power counting.[6] Therefore, after fixing a definite form of our lattice action in later sections, we confirm that such an "exotic" does not occur with our lattice action, by using the Reisz power counting theorem [14] for lattice Feynman integrals.

Accepting temporarily the above argument based on the continuum power counting, it is clear that a lattice action for the $\mathcal{N} = (2, 2)$ SYM theory in 2 dimensions (in our scenario) is large extent arbitrary. (A coefficient of mass counter terms of course depends on a lattice action chosen.) By using this wide freedom, we can avoid the above problem

[5] A similar idea for lower dimensional SYM theories can be found in the discussion of ref. [13].

[6] For example, to a lattice action of a 2 dimensional $g\phi^3$ theory, one may add an "irrelevant" interaction such as $\sqrt{g}a\phi\partial_\mu\phi\partial_\mu\phi$ which induces a tadpole of $O(\sqrt{g}/a)$ and two point function of $O(ga^0)$; these spoil the continuum loop expansion and the continuum power counting.

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of complex fermion determinant. For definiteness, we start with a lattice formulation of the $\mathcal{N}=1$ SYM theory in 4 dimensions, in which the plaquette action and the Wilson- or the overlap-Dirac operator [15] are used. Then by a dimensional reduction, we obtain a lattice action for the $\mathcal{N}=(2,2)$ SYM theory in 2 dimensions. In this construction, the fermion determinant, even with presence of gauge and scalar fields, is real and, moreover, when the overlap-Dirac operator is used, semi-positive definite.[7] Another bonus of this construction is that scalar fields in the 2 dimensional theory, which are originally gauge fields along the reduced dimensions, are compact.[8] Thus there is no subtlety associated with an integration along the flat directions for which the classical potential energy vanishes. A non-compactness of scalar fields in the target theory is restored in the continuum limit.[9] These features of our formulation must be desirable for practical numerical simulations.

The above argument crucially depends on perturbation theory and we ignored a possible subtlety associated with the infrared (IR) divergences in this 2 dimensional massless theory. For the first point, we have no further comment and simply assume a validity of perturbation theory in a weak coupling phase. Note that in the present model, a dimensionless coupling constant ag_0 goes to zero in the continuum limit. On the second point, a careful treatment of zero modes will show that our program proceeds as expected.

This paper is organized as follows: In section 2, we summarize basic facts concerning the dimensional reduction and the $\mathcal{N}=(2,2)$ SYM theory in 2 dimensions, in particular on the one-loop effective potential in a finite box. In section 3, our lattice construction is presented. By comparing the one-loop effective potential in our lattice framework with that of the continuum theory, we determine a coefficient of required scalar mass counter terms. It is found that the coefficient is IR as well as UV finite. In section 4, we discuss further prospects and possible generalizations.

2. Target continuum theory

2.1 Dimensional reduction and the $\mathcal{N}=(2,2)$ SYM in 2 dimensions

The $\mathcal{N}=1$ SYM theory takes a particularly simple form when the spacetime dimension d is 4, 6 and 10 [19]. For example, the on-shell fields of the $\mathcal{N}=1$ SYM theory in 4 dimensions consist of the gauge boson and the adjoint Majorana fermion. Moreover, by applying the dimensional reduction [20] to this theory in $d=4$, the classical action of the $\mathcal{N}=2$ SYM theory in $d=3$, the $\mathcal{N}=(2,2)$ SYM theory in $d=2$ and the $\mathcal{N}=4$ "SYM" theory in $d=1$ is obtained [19].

[7]The domain-wall fermion [16] with an infinite 5 dimensional extent shares this feature and may be adopted in our formulation as well. We will not compute a corresponding coefficient of mass counter terms however.

[8]It is interesting to note that if all bosonic fields including scalar fields are compact in a lattice formulation with an *exact* nilpotent supersymmetry, then the Neuberger no go theorem [17] on a lattice BRS symmetry would imply a vanishing Witten index [18]. H.S. would like to thank Fumihiko Sugino for reminding this point.

[9]There is however a subtle issue whether lattice formulations based on non-compact scalar fields and compact scalar fields are in the same universality class; we do not consider this issue in the present paper.

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A similar statement holds in a spacetime with the euclidean signature, although in the 4 dimensional euclidean space the Majorana condition cannot be imposed in an SO(4) invariant way. The action of the $\mathcal{N}=1$ SU(N_c) SYM theory in $d=4$ euclidean space would be[10]

$$S=\int \mathrm{d}^4x\left\{\frac{1}{4}F^a_{MN}F^a_{MN}+\frac{1}{2}\lambda^{aT}C\Gamma_M D_M\lambda^a\right\}, \tag{2.1}$$

where

$$F^a_{MN}=\partial_M A^a_N-\partial_N A^a_M+g_0 f_{abc}A^b_M A^c_N\,,\qquad D_M\lambda^a=\partial_M\lambda^a+g_0 f_{abc}A^b_M\lambda^c\,, \tag{2.2}$$

with g_0 being the dimensionless gauge coupling constant. In eq. (2.1), the matrix C is a charge conjugation matrix such that[11]

$$C\Gamma_M C^{-1}=-\Gamma_M^T\,,\qquad C\Gamma C^{-1}=\Gamma^T\,,\qquad C^{-1}=C^\dagger\,,\qquad C^T=-C\,. \tag{2.3}$$

Eq. (2.1) is a Wick rotated version of the $\mathcal{N}=1$ SYM theory in $d=4$ Minkowski spacetime. The above prescription for the Majorana fermion in the euclidean space [22, 23] can be understood also from a view point of the “Majorana decomposition” which precisely gives rise to the “half” the Dirac fermion in the euclidean space [21].

The action (2.1) is invariant under the super transformation

$$\begin{aligned}\delta_\xi A^a_M&=\lambda^{aT}C\Gamma_M\xi=-\xi^T C\Gamma_M\lambda^a\,,\\ \delta_\xi\lambda^a&=\frac{1}{2}F^a_{MN}\Sigma_{MN}\xi\,,\qquad \Sigma_{MN}=\frac{1}{2}[\Gamma_M,\Gamma_N]\,,\end{aligned} \tag{2.4}$$

owing to the Bianchi identity and the relation $\Gamma_M\Gamma_N\Gamma_R=(1/3!)\Gamma_{[M}\Gamma_N\Gamma_{R]}+\delta_{MN}\Gamma_R-\delta_{MR}\Gamma_N+\delta_{NR}\Gamma_M$. In addition to this symmetry, eq. (2.1) possesses the global $\mathrm{U}(1)_R$ symmetry

$$\delta_\epsilon A^a_M=0\,,\qquad \delta_\epsilon\lambda^a=i\epsilon\Gamma\lambda^a\,, \tag{2.5}$$

though this symmetry is broken by the anomaly.

By applying a dimensional reduction to eq. (2.1), one can deduce a euclidean version of the $\mathcal{N}=(2,2)$ SYM theory in 2 dimensions. The dimensional reduction amounts to set $\partial_3\Rightarrow 0$ and $\partial_2\Rightarrow 0$. To obtain a canonical normalization of fields, we also rescale the gauge potentials and the gauge coupling as $\ell A^a_M\Rightarrow A^a_M$ and $g_0/\ell\Rightarrow g_0$, by using a scale of length ℓ which may be regarded as a size of the reduced (or more appropriately compactified) directions. Then we regard $M=3$ and $M=2$ components of gauge potentials

[10] Our convention: Anti-hermitian generators T^a of SU(N_c) are normalized as $\mathrm{tr}\{T^aT^b\}=-(1/2)\delta_{ab}$. The totally anti-symmetric structure constants are defined by $[T^a,T^b]=f_{abc}T^c$ and $f_{abc}f_{abd}=N_c\delta_{cd}$ for SU(N_c). The capital Roman indices, M, N, … run over 0, 1, 2 and 3, and the Greek indices, μ, ν, … run over 0 and 1. A summation over repeated indices is understood unless noted otherwise. Dirac matrices for $d=4$ (denoted by Γ_M) and for $d=2$ (denoted by γ_μ) are hermitian and obey $\{\Gamma_M,\Gamma_N\}=2\delta_{MN}$ and $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$. The chiral matrices are defined by $\Gamma=-\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ and $\gamma=-i\gamma_0\gamma_1$. $\hat{M}$ and $\hat{\mu}$ denote a unit vector for the M direction and the μ direction, respectively.

[11] This is the matrix B_2 of ref. [21].

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as scalar fields as $A_3 \Rightarrow \phi$ and $A_2 \Rightarrow \varphi$. The variable λ is mapped to the Dirac fermion in 2 dimensions.[12] After this dimensional reduction, fermion bi-linears are mapped to

$$\begin{aligned}
&\lambda^{aT} C\Gamma_\mu \mathcal{O}_{ab}\lambda^b \Rightarrow 2\overline{\psi}^a \gamma_\mu \mathcal{O}_{ab}\psi^b \qquad \text{for } \mu = 0,\, 1\,,\\
&\lambda^{aT} C\Gamma_2 \mathcal{O}_{ab}\lambda^b \Rightarrow 2\overline{\psi}^a \gamma \mathcal{O}_{ab}\psi^b\,,\\
&\lambda^{aT} C\Gamma_3 \mathcal{O}_{ab}\lambda^b \Rightarrow 2\overline{\psi}^a (-i) \mathcal{O}_{ab}\psi^b\,,
\end{aligned} \tag{2.6}$$

where $\mathcal{O}$ is any anti-symmetric matrix with gauge and space indices. In this way, we have a euclidean version of the $\mathcal{N} = (2, 2)$ SYM theory in 2 dimensions:

$$\begin{aligned}
S = \int \mathrm{d}^2x \Big\{ &\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} D_\mu \phi^a D_\mu \phi^a + \frac{1}{2} D_\mu \varphi^a D_\mu \varphi^a + \frac{1}{2} g_0^2 f_{abc} f_{ade} \varphi^b \phi^c \varphi^d \phi^e + \\
&+ \overline{\psi}^a \gamma_\mu D_\mu \psi^a - i g_0 f_{abc} \overline{\psi}^a (\phi^b + i\gamma\varphi^b)\psi^c \Big\}.
\end{aligned} \tag{2.7}$$

This action is invariant under the super transformation

$$\begin{aligned}
\delta_\theta A^a_\mu &= \overline{\psi}^a \gamma_\mu \theta - \overline{\theta}\gamma_\mu \psi^a\,,\\
\delta_\theta \phi^a &= -i\overline{\psi}^a \theta + i\overline{\theta}\psi^a\,, \qquad \delta_\theta \varphi^a = \overline{\psi}^a \gamma\theta - \overline{\theta}\gamma\psi^a\,,\\
\delta_\theta \psi^a &= \frac{1}{2} F^a_{\mu\nu}\sigma_{\mu\nu}\theta - i\gamma_\mu D_\mu(\phi^a + i\gamma\varphi^a)\theta + i g_0 f_{abc}\varphi^b\phi^c\gamma\theta\,,\\
\delta_\theta \overline{\psi}^a &= -\frac{1}{2}\overline{\theta}\sigma_{\mu\nu} F^a_{\mu\nu} - i\overline{\theta}\gamma_\mu D_\mu(\phi^a - i\gamma\varphi^a) + i g_0 f_{abc}\overline{\theta}\gamma\varphi^b\phi^c\,,
\end{aligned} \tag{2.8}$$

(where $\sigma_{\mu\nu} = (1/2)[\gamma_\mu, \gamma_\nu]$; one notes that $\sigma_{\mu\nu} = i\epsilon_{\mu\nu}\gamma$ with $\epsilon_{01} = 1$ and $\gamma\gamma_\mu = i\epsilon_{\mu\nu}\gamma_\nu$) which can be obtained by applying the dimensional reduction to eq. (2.4). Note that in eq. (2.7), ψ^a and $\overline{\psi}^a$ are regarded independent variables. The $\mathrm{U}(1)_R$ symmetry (2.5) becomes the fermion number symmetry

$$\begin{aligned}
&\delta_\epsilon A^a_\mu = 0\,, \qquad \delta_\epsilon \phi^a = 0\,, \qquad \delta_\epsilon \varphi^a = 0\,,\\
&\delta_\epsilon \psi^a = -i\epsilon\psi^a\,, \qquad \delta_\epsilon \overline{\psi}^a = i\epsilon\overline{\psi}^a\,,
\end{aligned} \tag{2.9}$$

and, on the other hand, the rotational symmetry in the 2-3 plane becomes the internal chiral symmetry

$$\begin{aligned}
&\delta_\epsilon A^a_\mu = 0\,, \qquad \delta_\epsilon \phi^a = 2\epsilon\varphi^a\,, \qquad \delta_\epsilon \varphi^a = -2\epsilon\phi^a\,,\\
&\delta_\epsilon \psi^a = i\epsilon\gamma\psi^a\,, \qquad \delta_\epsilon \overline{\psi}^a = i\epsilon\overline{\psi}^a\gamma\,,
\end{aligned} \tag{2.10}$$

after the dimensional reduction.

[12] In a representation in $d = 4$ in which $\Gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\Gamma_i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}$, the Dirac field in $d = 2$ can be defined from components of the Majorana field by $\psi = \ell\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $\overline{\psi} = \ell(-\lambda_4, -\lambda_3)$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The Dirac matrices in $d = 2$ then take the form, $\gamma_0 = \sigma^3$, $\gamma_1 = \sigma^2$ and $\gamma = -\sigma^1$ and $B = \sigma^2$.

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2.2 One-loop effective potential in the continuum theory

As stated in Introduction, we correct a breaking of the supersymmetry in our lattice formulation by supplementing scalar mass counter terms. To find an appropriate value of a coefficient of counter terms, here we compute the one-loop effective potential for scalar fields in the $\mathcal{N} = (2, 2)$ SYM theory in 2 dimensions.[13] For this perturbative calculation, we add the gauge fixing term

$$S_{\rm gf} = \int {\rm d}^2x\, \frac{1}{2}\lambda_0 \partial_\mu A_\mu^a \partial_\nu A_\nu^a\,, \tag{2.11}$$

to the action (2.7). With this gauge fixing term, the Faddeev-Popov ghosts couple only to gauge potentials and the ghosts are irrelevant to the present calculation of the one-loop effective potential for scalar fields.

Perturbation theory in the present model, a massless theory in 2 dimensions, is full of IR divergences and we need a careful treatment of zero modes. For a reliable treatment of zero modes, we define the theory in a finite box with size L. We further impose the periodic boundary conditions for all fields. The periodic boundary condition is consistent with the super transformation (2.8) and invariance of the action (2.7). Then the momentum becomes discrete and is given by

$$p_\mu = \frac{2\pi}{L} n_\mu\,, \qquad n_\mu \in \mathbb{Z}\,. \tag{2.12}$$

As usual, the one-loop effective potential for scalar fields is obtained by performing gaussian integrations over fluctuations around the expectation value of scalar fields. So we set

$$\begin{aligned}
A_\mu^a(x) &= \sum_p \frac{1}{L} e^{ipx} \tilde{A}_\mu^a(p)\,,\\
\phi^a(x) &= \phi^a + \sum_p \frac{1}{L} e^{ipx} \tilde{\phi}^a(p)\,, \qquad \varphi^a(x) = \varphi^a + \sum_p \frac{1}{L} e^{ipx} \tilde{\varphi}^a(p)\,,\\
\psi^a(x) &= \sum_p \frac{1}{L} e^{ipx} \tilde{\psi}^a(p)\,, \qquad \overline{\psi}^a(x) = \sum_p \frac{1}{L} e^{ipx} \tilde{\overline{\psi}}^a(p)\,,
\end{aligned} \tag{2.13}$$

where expectation values ϕ^a and φ^a are taken to be constant. Substituting these into the action, (2.7) plus (2.11) and picking out terms quadratic in fluctuations, we have

$$\begin{aligned}
S + S_{\rm gf} = \frac{1}{2}\sum_p \Big\{ & \tilde{A}_\mu(-p)[\delta_{\mu\nu}p^2 - (1-\lambda_0)p_\mu p_\nu - \delta_{\mu\nu}(\Phi^2 + \Psi^2)]\tilde{A}_\nu(p) + \\
& + \tilde{\phi}(-p)(p^2 - \Psi^2)\tilde{\phi}(p) + \tilde{\varphi}(-p)(p^2 - \Phi^2)\tilde{\varphi}(p) + \\
& + \tilde{A}_\mu(-p) i p_\mu \Phi \tilde{\phi}(p) + \tilde{\phi}(-p) i p_\mu \Phi \tilde{A}_\mu(p) + \\
& + \tilde{A}_\mu(-p) i p_\mu \Psi \tilde{\varphi}(p) + \tilde{\varphi}(-p) i p_\mu \Psi \tilde{A}_\mu(p) + \\
& + \tilde{\phi}(-p)(2\Psi\Phi - \Phi\Psi)\tilde{\varphi}(p) + \tilde{\varphi}(-p)(2\Phi\Psi - \Psi\Phi)\tilde{\phi}(p) + \\
& + 2\tilde{\overline{\psi}}(-p)(\gamma_\mu i p_\mu - i\Phi + \gamma\Psi)\tilde{\psi}(p) \Big\} + \cdots\,,
\end{aligned} \tag{2.14}$$

[13] For this purpose, supersymmetric Ward-Takahashi identities would be used as well. We found that, however, an examination of the effective potential is much simpler.

where we have introduced matrices

$$(\Phi)_{ab} = g_0 f_{acb}\phi^c\,, \qquad (\Psi)_{ab} = g_0 f_{acb}\varphi^c\,, \tag{2.15}$$

and abbreviated contractions in group indices. Gaussian integrations with respect to fluctuations are straightforward and we begin with integrations over zero-modes.

Gaussian integrations with respect to fluctuations with $p = 0$ (zero modes) give rise to the following contribution to the effective potential

$$\frac{1}{L^2}\Bigg\{\mathrm{tr}\log\{\Phi^2+\Psi^2\}+\frac{1}{2}\,\mathrm{tr}\log\begin{pmatrix}\Psi^2 & -2\Psi\Phi+\Phi\Psi\\ -2\Phi\Psi+\Psi\Phi & \Phi^2\end{pmatrix}-$$
$$-\,\mathrm{tr}\log\{\Phi^2+\Psi^2+i(\Psi\Phi-\Phi\Psi)\}\Bigg\}\,, \tag{2.16}$$

where tr denotes the trace with respect to group indices. In this expression, the first line is a contribution from bosonic zero modes and the second line comes from fermionic zero modes. One would expect that the above three terms cancel out but this is not the case. This becomes clear by considering configurations with $[\Psi,\Phi] = 0$ or equivalently $f_{abc}\varphi^b\phi^c = 0$. These are nothing but configurations in the flat directions along which the classical potential energy vanishes. For these configurations, the first line of eq. (2.16) becomes singular as $\log 0$ and the second line remains regular. Thus three terms in eq. (2.16) do not cancel even at minima of the classical potential.

The non-zero radiative corrections in the one-loop effective potential (2.16) are *not* in contradiction with a general property of supersymmetric theories that the vacuum energy vanishes when the supersymmetry is not spontaneously broken.[14] The effective potential coincides with the vacuum energy only for minima of the effective potential (because the external sources vanish only at minima of the effective potential). Moreover, in our present case, at minima of the classical potential, namely at configurations along the flat directions, the quadratic term of bosonic zero modes acquires zero eigenfunctions and the loop expansion (or the $\hbar$ expansion) breaks down.[15] To see that the flat directions actually do not receive any radiative corrections, one has to consider an integration over zero modes with a full part of the action, not only the quadratic part.[16] In any case, irrespective of one's interpretation on the "one-loop" radiative corrections from zero modes (2.16), it is a very property of the target *continuum* theory in a finite box and should be reproduced by any sensible lattice formulation. We note that the correction (2.16) vanishes and a naive expectation is reproduced in the $L \to \infty$ limit.

[14] In our present problem, this property can be shown in the following way, for example: After introducing the auxiliary field, the euclidean action can be expressed as a super transformation of a certain gauge invariant operator. Assuming that the supersymmetry is not spontaneously broken, this shows that the vacuum energy is independent of the gauge coupling and may be adjusted to be zero. The gauge fixing term and the Faddeev-Popov ghost term do not contribute to the vacuum energy owing to the Slavnov-Taylor identity.

[15] It would thus be inappropriate to call eq. (2.16) as the "one loop" effective potential for configurations along the flat directions.

[16] This kind of study can be found in ref. [8].

We now turn to gaussian integrations over fluctuations with $p \neq 0$. For a contribution of these non-zero modes, it is possible to expand the effective potential with respect to expectations values. In the quadratic order of ϕ^a and φ^a which will be relevant for later discussions, we have

$$\frac{1}{L^2}\left\{-\sum_{p\neq 0}\frac{1}{p^2}+\sum_{p\neq 0}\frac{1}{p^2}\right\}\mathrm{tr}\{\Phi^2+\Psi^2\}=\frac{1}{L^2}\left\{\sum_{p\neq 0}\frac{1}{p^2}-\sum_{p\neq 0}\frac{1}{p^2}\right\}g_0^2N_c(\phi^a\phi^a+\varphi^a\varphi^a)\,, \tag{2.17}$$

where the first and the second terms in the parentheses come from bosonic and fermionic fluctuations, respectively. Thus if we apply a uniform UV regularization for bosonic modes and for fermionic modes, then a total contribution to the effective potential vanishes. Obviously, this cancellation is a result of an underlying supersymmetry. Note that this result is independent of the gauge parameter λ_0.

In summary, the one-loop effective potential in the $\mathcal{N}=(2,2)$ SYM theory in 2 dimensions defined in a box of size L with the periodic boundary conditions possesses the following properties. (1) Contributions from zero modes do not cancel out and take the form (2.16). (2) As eq. (2.17) shows, contributions from non-zero modes to the quadratic parts precisely cancel out under a supersymmetric UV regularization. In the next section, we determine a coefficient of mass counter terms in our lattice formulation so that these properties of the target theory are reproduced in the continuum limit.

3. Lattice formulation of the $\mathcal{N}=(2,2)$ SYM theory in 2 dimensions

3.1 In the case of the Wilson fermion

Our lattice action consists of three parts:

$$S[U,\lambda]=S_{\mathrm{G}}[U]+S_{\mathrm{F}}[U,\lambda]+S_{\mathrm{counter}}[U]\,. \tag{3.1}$$

For bosonic fields, we use the standard plaquette action

$$\begin{aligned} S_{\mathrm{G}}[U] &= \frac{1}{a^2g_0^2}\sum_{x\in\Gamma}\sum_{M,N}\mathrm{Re}\,\mathrm{tr}\{1-P(x,M,N)\}\,,\\ P(x,M,N) &= U(x,M)U(x+a\hat{M},N)U(x+a\hat{N},M)^{-1}U(x,N)^{-1}\,, \end{aligned} \tag{3.2}$$

where $U(x,M)\in\mathrm{SU}(N_c)$ represents the link variable. For the fermion sector, we use the Wilson-Dirac operator D_{w}

$$S_{\mathrm{F}}[U,\lambda]=-a^2\sum_{x\in\Gamma}\mathrm{tr}\{\lambda(x)CD_{\mathrm{w}}\lambda(x)\}\,,\qquad D_{\mathrm{w}}=\frac{1}{2}\{\Gamma_M(\nabla_M^*+\nabla_M)-a\nabla_M^*\nabla_M\}\,, \tag{3.3}$$

with covariant differences for the adjoint representation

$$\begin{aligned} \nabla_M\lambda(x) &= \frac{1}{a}\left\{U(x,M)\lambda(x+a\hat{M})U(x,M)^{-1}-\lambda(x)\right\},\\ \nabla_M^*\lambda(x) &= \frac{1}{a}\left\{\lambda(x)-U(x-a\hat{M},M)^{-1}\lambda(x-a\hat{M})U(x-a\hat{M},M)\right\}. \end{aligned} \tag{3.4}$$

We use the overlap-Dirac operator in the next subsection. The last term in eq. (3.1) is a mass counter term which will be specified below.

One verifies that by setting

$$U(x, M) = \exp\{a g_0 A_M^a(x) T^a\}\,, \tag{3.5}$$

the classical continuum limit $a \to 0$ of eq. (3.1) without S_{counter} is nothing but eq. (2.1), the $\mathcal{N} = 1$ SYM theory in $d = 4$ dimensions, except overall powers of a. In writing eq. (3.1), we already performed the rescaling associated to the dimensional reduction $d = 4 \to d = 2$ by identifying $\ell = a$. So the gauge coupling g_0 in this section has a dimension of mass.

To realize a dimensional reduction from $d = 4$ to $d = 2$, we set the following boundary conditions

$$\begin{aligned} U(x + a\hat{M}, N) &= U(x, N)\,, & \lambda(x + a\hat{M}) &= \lambda(x) & \text{for } M &= 2, 3\,, \\ U(x + L\hat{M}, N) &= U(x, N)\,, & \lambda(x + L\hat{M}) &= \lambda(x) & \text{for } M &= 0, 1\,. \end{aligned} \tag{3.6}$$

Namely, we reduce (or compactify) directions of $M = 2$ and $M = 3$ and impose the periodic boundary conditions for other two directions. The size of the 2 dimensional box L is assumed to be an integer-multiple of the lattice spacing a. Thus our 2 dimensional lattice is given by

$$\Gamma = \left\{x \in a\mathbb{Z}^2 \mid 0 \leq x_\mu < L\right\}. \tag{3.7}$$

The link variables are integrated with the invariant Haar measure $\prod_{x\in\Gamma} \prod_M \mathrm{d}\mu(U(x, M))$ as usual.

Scalar fields in the $\mathcal{N} = (2, 2)$ SYM theory in 2 dimensions are identified with gauge potentials in $M = 3$ and $M = 2$ directions:

$$U(x, 3) = \exp\{a g_0 \phi^a(x) T^a\}\,, \qquad U(x, 2) = \exp\{a g_0 \varphi^a(x) T^a\}\,. \tag{3.8}$$

The classical continuum limit of eq. (3.1) (without S_{counter}) with the boundary conditions (3.6) reproduces the $\mathcal{N} = (2, 2)$ SYM theory in 2 dimensions (2.7). In our lattice construction based on the dimensional reduction, scalar fields in the $\mathcal{N} = (2, 2)$ SYM theory in 2 dimensions are given by components of link variables as eq. (3.8). Therefore an integration along these degrees of freedom is compact. In particular, the flat directions in the target theory, along which the potential term $(f_{abc}\varphi^b\phi^c)^2$ vanishes, become compact for finite lattice spacings. Thus no subtlety is expected for numerical integrations along these flat directions. A non-compactness of scalar fields in the target theory is, as gauge potentials, restored in the continuum limit $a \to 0$.

Our proposal is similar to that of ref. [24] at the point that an extended SYM theory is formulated as a dimensional reduction of a lattice formulation of the $\mathcal{N} = 1$ SYM theory in a higher dimension. Contrary to ref. [24], however, we do not claim no need of tuning in a resulting low dimensional lattice theory. It is true that $\mathcal{N} = 1$ SYM theory in 4 dimensions [25], for example, when formulated with the overlap-Dirac operator or with the domain-wall, requires no fine tuning for a supersymmetric continuum limit [26], owing to the exact lattice chiral symmetry. After a dimensional reduction, however, a rotational symmetry among reduced and un-reduced directions is violated and scalar mass terms are not prohibited in general. In our formulation, this breaking of supersymmetry is corrected by supplementing scalar mass terms.

In the continuum theory, a mixed mass term $\varphi^a\phi^a$ is forbidden by the chiral symmetry (2.10) and only a symmetric mass term of the form $\phi^a\phi^a+\varphi^a\varphi^a$ is allowed (when the supersymmetry is ignored). This persists in our lattice theory, owing to the exact discrete symmetry

$$\begin{gathered}U(x,0)\to U(x,0)\,,\quad U(x,1)\to U(x,1)\,,\quad U(x,2)\to U(x,3)^{-1}\,,\quad U(x,3)\to U(x,2)\,,\\ \lambda(x)\to\exp\left\{-\frac{\pi}{4}\Sigma_{23}\right\}\lambda(x)\,,\end{gathered}\tag{3.9}$$

which is a lattice analogue of the chiral rotation (2.10) with $\pi/4$ radian (recall that the chiral rotation was originally a space rotation in the 2-3 plane). The scalar mass counter terms thus may be taken as

$$S_{\text{counter}}[U]=-\mathcal{C}N_c\sum_{x\in\Gamma}\left(\operatorname{tr}\{U(x,3)+U(x,3)^{-1}-2\}+\operatorname{tr}\{U(x,2)+U(x,2)^{-1}-2\}\right),\tag{3.10}$$

as this combination reduces to the symmetric mass term in the classical continuum limit. Our task is therefore to determine an appropriate coefficient $\mathcal{C}$.

To determine $\mathcal{C}$, we compute the one-loop effective potential of scalar fields and compare it with that in the target theory in section 2.2. For this perturbative calculation, we add the following gauge fixing term[17]

$$S_{\text{gf}}[U]=-a^2\sum_{x\in\Gamma}\sum_{\mu,\nu=0}^{1}\lambda_0\operatorname{tr}\{\partial_\mu^*A_\mu(x)\partial_\nu^*A_\nu(x)\}\,,\tag{3.11}$$

to the lattice action (3.1), where λ_0 being the gauge parameter. As in the continuum theory, the ghost fields do not contribute to the one-loop effective potential of scalar fields.

We also have to take account of the jacobian from the invariant group measure $\prod_{x\in\Gamma}\times\prod_M \mathrm{d}\mu(U(x,M))$ for link variables to a linear measure which is used in perturbation theory. That is given by [27]

$$\begin{aligned}\prod_{x\in\Gamma}\prod_{M}\mathrm{d}\mu(U(x,M))&=\prod_{x\in\Gamma}\prod_{M,a}\mathrm{d}A_M^a\,e^{-S_{\text{measure}}},\\ S_{\text{measure}}&=-\sum_{x\in\Gamma}\sum_{M}\frac{1}{2}\operatorname{tr}\ln\left\{2\frac{\cosh(a\mathcal{A}_M(x))-1}{a\mathcal{A}_M(x)}\right\},\end{aligned}\tag{3.12}$$

where $\mathcal{A}_M$ is the gauge potential in the adjoint representation, $(\mathcal{A}_M)_{ab}=g_0f_{acb}A_M^c$, and tr denotes the trace over group indices. This factor gives rise to mass terms of scalar fields,

$$S_{\text{measure}}=a^2\sum_{x\in\Gamma}\left\{\frac{1}{24}g_0^2N_c[\phi^a(x)\phi^a(x)+\varphi^a(x)\varphi^a(x)]+\cdots\right\},\tag{3.13}$$

that should also be included in the *one-loop* effective potential, because this term is $O(g_0^2)$.

[17] $\partial_\mu f(x)=(1/a)\{f(x+a\hat{\mu})-f(x)\}$ and $\partial_\mu^* f(x)=(1/a)\{f(x)-f(x-a\hat{\mu})\}$ denote the forward and the backward differences, respectively.

Now, to compute one-loop radiative corrections to the effective potential of scalar fields, we substitute an expansion similar to eq. (2.13) into the lattice action $S_{\rm G}+S_{\rm F}+S_{\rm gf}$, with understandings (3.5) and (3.8). In the present lattice case, the momentum p is still discrete $p_\mu = (2\pi/L)n_\mu$ but is limited within the Brillouin zone

$$\mathcal{B} = \left\{ p \in \mathbb{R}^2 \mid |p_\mu| \le \frac{\pi}{a} \right\}. \tag{3.14}$$

We pick out terms quadratic in fluctuations and perform gaussian integrations over fluctuations.

Let us first consider an integration over zero modes. A form of the lattice action quadratic in zero modes is in general different from that in the continuum (2.14) by terms of $O(a)$. An integration over zero modes in our lattice theory thus would give a different effective potential from eq. (2.16). However, the difference is $O(a)$, because no UV divergence arises from an integration over zero modes (these are finite degrees of freedom). As a result, in the continuum limit $a \to 0$, a contribution of zero modes to the effective potential coincides with eq. (2.16).

Next, we consider non-zero modes. The action quadratic in fluctuations is

$$\begin{aligned} S_{\rm G}+S_{\rm F}+S_{\rm gf} = \frac{1}{2}\sum_p \Bigg\{ \sum_{\mu,\nu} \tilde{A}_\mu(-p) \Bigg[& \delta_{\mu\nu}\hat{p}^2 + (1-\lambda_0)\frac{1}{a^2}(e^{iap_\mu}-1)(1-e^{-iap_\nu}) - \\ & - \delta_{\mu\nu}\Phi^2 \Bigg] \tilde{A}_\nu(p) + \\ & + \tilde{\phi}(-p)\left(\hat{p}^2 + \frac{1}{12}a^2\hat{p}^2\Phi^2\right)\tilde{\phi}(p) + \tilde{\varphi}(-p)(\hat{p}^2-\Phi^2)\tilde{\varphi}(p) + \\ & + \sum_\mu \tilde{A}_\mu(-p)\frac{1}{a}(e^{ip_\mu a}-1)\Phi\tilde{\phi}(p) + \\ & + \sum_\mu \tilde{\phi}(-p)\frac{1}{a}(1-e^{-ip_\mu a})\Phi\tilde{A}_\mu(p) + \\ & + \tilde{\lambda}(-p)C(\tilde{D}_{\rm w}^{(0)}(p) + \Gamma_3\Phi - \frac{1}{2}a\Phi^2)\tilde{\lambda}(p) \Bigg\} + \cdots, \end{aligned} \tag{3.15}$$

where we have retained only terms relevant to a quadratic term in the effective potential of ϕ. In the above expression, $\tilde{D}_{\rm w}^{(0)}(p)$ is the momentum representation of the free Wilson-Dirac operator

$$\tilde{D}_{\rm w}^{(0)}(p) = i\Gamma_\mu \mathring{p}_\mu + \frac{1}{2}a\hat{p}^2, \tag{3.16}$$

and $\mathring{p}_\mu$ and $\hat{p}_\mu$ denote momentum variables defined by

$$\begin{aligned} \mathring{p}_\mu &= \frac{1}{a}\sin(ap_\mu), \qquad \hat{p}_\mu = \frac{2}{a}\sin\left(\frac{1}{2}ap_\mu\right), \\ \mathring{p}^2 &= \sum_{\mu=0}^{1}\mathring{p}_\mu^2, \qquad \hat{p}^2 = \sum_{\mu=0}^{1}\hat{p}_\mu^2. \end{aligned} \tag{3.17}$$

By comparing eq. (3.15) with eq. (2.14), we see various form of lattice artifacts.

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The gaussian integrations over non-zero mode fluctuations are straightforward and, including a contribution of the measure term (3.13), we have

$$\frac{1}{L^2}\sum_{p\neq 0}\left(\frac{1}{\hat{p}^2}-\frac{1+\frac{1}{2}a^2\hat{p}^2}{\mathring{p}^2+\frac{a^2}{4}(\hat{p}^2)^2}\right)g_0^2N_c\phi^a\phi^a\equiv-\frac{1}{2}\mathcal{C}g_0^2N_c\phi^a\phi^a\,, \tag{3.18}$$

as the effective potential. The first term in the parentheses is a contribution of bosonic fields and the second is a contribution of the (Wilson) fermion. Note that this expression is independent of the gauge parameter λ_0. Comparing this with eq. (2.17), we see that a cancellation of a bosons' contribution and a fermions' contribution is not perfect owing to lattice artifacts. We correct this deviation from the target continuum theory by adding the scalar mass counter terms (3.10). An important observation is that eq. (3.18) is an IR as well as UV finite quantity. The coefficient $\mathcal{C}$ is a dimensionless number that depends only on the ratio a/L. In the continuum limit $a\to 0$, therefore, the summation can be replaced by an integral

$$\frac{1}{L^2}\sum_{p\neq 0}\to\int_{\mathcal{B}}\frac{\mathrm{d}^2p}{(2\pi)^2}\,, \tag{3.19}$$

and we have

$$\mathcal{C}=-2\int_{-\pi}^{\pi}\frac{\mathrm{d}^2p}{(2\pi)^2}\left(\frac{1}{\hat{p}^2}-\frac{1+\frac{1}{2}\hat{p}^2}{\mathring{p}^2+\frac{1}{4}(\hat{p}^2)^2}\right)=0.65948255(8) \tag{3.20}$$

where we have rescaled the integration variables as $p_\mu\to p_\mu/a$ and changed the definition of momentum variables as $\mathring{p}_\mu=\sin(p_\mu)$ and $\hat{p}_\mu=2\sin(p_\mu/2)$. This completes our lattice formulation of the $\mathcal{N}=(2,2)$ SYM theory in 2 dimensions which uses the Wilson-Dirac operator. Namely, we claim that, after including the mass counter terms (3.10) with the coefficient (3.20), the target theory is obtained in the continuum limit *without any further tuning of parameters*. We finally remark that the lattice formulation in this subsection allows the strong coupling expansion.

3.2 In the case of the overlap fermion

A use of the overlap fermion in our framework has a great practical advantage because the fermion determinant is real and moreover *semi-positive*. In this case, the fermion part of our lattice action is given by

$$S_{\mathrm{F}}[U,\lambda]=-a^2\sum_{x\in\Gamma}\mathrm{tr}\{\lambda(x)CD\lambda(x)\}\,, \tag{3.21}$$

where the overlap-Dirac operator D is defined by

$$D=\frac{1}{a}\{1-A(A^\dagger A)^{-1/2}\}\,,\qquad A=1-aD_{\mathrm{w}}\,, \tag{3.22}$$

from the Wilson-Dirac operator (3.3). As shown in ref. [28], then the fermion determinant, or more precisely the pfaffian, is semi-positive definite (see also ref. [29]; for a proof of this fact from general grounds, see an appendix of ref. [21]). Since we formulate the $\mathcal{N}=(2,2)$

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SYM theory in 2 dimensions by a dimensional reduction of a 4 dimensional lattice gauge theory defined above, the fermion determinant in 2 dimensional sense is also real and semi-positive even with presence of scalar fields.

For the gauge sector, one may use the plaquette action (3.2), but from various point of view, the modified plaquette action [30 – 32]

$$
\begin{aligned}
S_{\mathrm{G}}[U] &= \frac{1}{a^2 g_0^2}\sum_{x\in\Gamma}\sum_{M,N}\mathcal{L}(x,M,N),\\
\mathcal{L}(x,M,N) &= \begin{cases} \dfrac{\mathrm{Re}\,\mathrm{tr}\{1-P(x,M,N)\}}{1-\mathrm{Re}\,\mathrm{tr}\{1-P(x,M,N)\}/\epsilon'} & \text{if } \mathrm{Re}\,\mathrm{tr}\{1-P(x,M,N)\}<\epsilon',\\ \infty & \text{otherwise,}\end{cases}
\end{aligned}
\tag{3.23}
$$

which dynamically imposes the admissibility [33, 34], is more appropriate.[18] Recall that, in our framework, the plaquette variable contains scalar fields in the $\mathcal{N}=(2,2)$ SYM theory as well. Thus the admissibility restricts also a configuration of scalar fields. One can confirm that, however, this way of modification of the gauge action does not affect the one-loop effective potential of the scalar field ϕ in the continuum limit (that we will compute). Thus for the gauge sector, we can use the identical result as eq. (3.18) (the first term).

For the fermion sector, we expand the action S_{F} around the expectation value to $O(\phi^2)$. Analogously to the last line of eq. (3.15), we have

$$
\begin{aligned}
S_{\mathrm{F}} = \frac{1}{2}\sum_p\Bigg\{\tilde\lambda(-p)C\bigg[&\tilde D^{(0)}(p) + \tilde X(p)^{-1/2}(\Gamma_3\Phi - \frac{1}{2}a\Phi^2)\\
&-\frac{1}{4}a^3\tilde X(p)^{-3/2}\tilde A^{(0)}(p)\hat p^2\Phi^2\bigg]\tilde\lambda(p)\Bigg\} + \cdots,
\end{aligned}
\tag{3.24}
$$

where

$$
\begin{aligned}
\tilde W(p) &= 1 - \frac{1}{2}a^2\hat p^2,\\
\tilde A^{(0)}(p) &= 1 - a\tilde D_{\mathrm{w}}^{(0)}(p) = \tilde W(p) - i\Gamma_\mu a\mathring{p}_\mu,\\
\tilde X(p) &= \tilde A^{(0)}(p)^\dagger A^{(0)}(p) = \tilde W(p)^2 + a^2\mathring{p}^2,\\
\tilde D^{(0)}(p) &= \frac{1}{a}\{1 - \tilde A^{(0)}(p)\tilde X(p)^{-1/2}\}\,.
\end{aligned}
\tag{3.25}
$$

As in the previous subsection, as a contribution of non-zero modes to the one-loop effective potential, we have

$$
\frac{1}{L^2}\sum_{p\neq0}\left\{\frac{1}{\hat p^2} - \frac{1}{\mathring{p}^2 + \frac{1}{a^2}(\tilde X^{1/2}-\tilde W)^2}\,\frac{\tilde W + a^2\mathring{p}^2}{\tilde X^{1/2}}\right\}g_0^2 N_c\phi^a\phi^a\,. \tag{3.26}
$$

[18] In our present case of the adjoint fermion, the overlap-Dirac operator is guaranteed to be well-defined if the admissibility $\|1-\mathcal{P}(x,M,N)\|<\epsilon$ for all x, M and N, where $\|A\|$ is the matrix norm, $\mathcal{P}^{ab} = -2\,\mathrm{tr}\{T^aPT^bP^{-1}\}$ denotes the plaquette in the adjoint representation and ϵ is a certain constant, holds. Because of the inequality $\|1-\mathcal{P}(x,M,N)\|\le 2\|1-P(x,M,N)\|\le 2\sqrt{2\,\mathrm{Re}\,\mathrm{tr}\{1-P(x,M,N)\}}$, the action (3.23) imposes this admissibility associated to the adjoint fermion by choosing $\epsilon'\le\epsilon^2/8$.

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Thus, in the limit $a \to 0$, the coefficient of the mass counter terms (3.10) is given by

$$\begin{aligned}\mathcal{C} &= -2\int_{-\pi}^{\pi} \frac{\mathrm{d}^2 p}{(2\pi)^2} \left\{ \frac{1}{\hat{p}^2} - \frac{1}{\mathring{p}^2 + (\tilde{X}^{1/2} - \tilde{W})^2} \frac{\tilde{W} + \mathring{p}^2}{\tilde{X}^{1/2}} \right\} \\ &= -0.28891909(1) \end{aligned} \tag{3.27}$$

where we have rescaled the integration variables as $p_\mu \to p_\mu/a$ and changed the definition of momentum variables as $\mathring{p}_\mu = \sin(p_\mu)$, $\hat{p}_\mu = 2\sin(p_\mu/2)$, $\tilde{W}(p) = 1 - \frac{1}{2}\hat{p}^2$ and $\tilde{X}(p) = \tilde{W}(p)^2 + \mathring{p}^2$. Thus, when the overlap-Dirac operator is used, a coefficient of the mass counter terms (3.10) is given by eq. (3.27).

3.3 UV power counting of lattice Feynman integrals

In this subsection, we return to our argument based on the continuum power counting in Introduction and confirm its validity. That is, with our lattice action, we show that only one-loop scalar two-point functions are potentially UV diverging. For this, we utilize the Reisz power counting theorem [14] on lattice Feynman integrals.

According to this theorem, the *superficial or overall* degree of UV divergence of a lattice Feynman integral

$$I_F = \int_{\mathcal{B}} \mathrm{d}^2 k_1 \cdots \mathrm{d}^2 k_\ell \, \frac{V(k, q; a)}{C(k, q; a)}\,, \tag{3.28}$$

where q collectively denotes the external momenta, associated to an ℓ-loop Feynman graph F, is given by

$$\deg I_F = 2\ell + \deg V - \deg C\,, \tag{3.29}$$

where $\deg V$ and $\deg C$ are the UV degree of the numerator V and the denominator C, respectively. The UV degree for the numerator, $\deg V$, is defined by the integer ν in the asymptotic behavior

$$V\left(\lambda k, q; \frac{a}{\lambda}\right) = K\lambda^\nu + O(\lambda^{\nu-1})\,, \tag{3.30}$$

for $\lambda \to \infty$ and the UV degree of the denominator $\deg C$ is similarly defined.

Now consider the Feynman rule resulting from our lattice action. From expressions (3.15) and (3.24), we immediately find that propagators of bosonic fields (i.e., the gauge potential and the scalar fields) contribute to $\deg C$ by 2 while the fermion propagators (with either choice of Dirac operators) contribute to $\deg C$ by 1. For interaction vertices, we find that purely bosonic b point vertices contribute to $\deg V$ by $4 - b$ at most and interaction vertices of the fermion with b bosonic lines $1 - b$ at most. From these, we have

$$\begin{aligned}\deg I_F &\leq 2\ell + \sum_k n_k(4 - b_k) + \sum_l \tilde{n}_l(1 - \tilde{b}_l) - 2i_b - i_f \\ &= 2 - \frac{1}{2} i_f - \sum_k n_k(b_k - 2) - \sum_l \tilde{n}_l \tilde{b}_l\,, \end{aligned} \tag{3.31}$$

where we have assumed that the diagram F contains n_k purely bosonic vertices with b_k boson lines and $\tilde{n}_l$ interaction vertices of the fermion with $\tilde{b}_l$ bosonic lines. i_b (i_f) denotes the number of boson (fermion) internal lines and e_b (e_f) denotes the number of boson (fermion) external lines of the diagram. From the first line to the second, we have used the "topological relations" $\ell = i_b + i_f - \{\sum_k n_k + \sum_l \tilde{n}_l - 1\}$ and $2\sum_l \tilde{n}_l = 2i_f + e_f$.

By noting that $b_k \geq 3$ and $\tilde{b}_l \geq 1$, we see that the total number of vertices, $\sum_k n_k + \sum_l \tilde{n}_l$, must be 1 or 2 for $\deg I_F$ to be non-negative. Then it is easy to see that, besides vacuum bubbles, only one-loop tadpoles (for which $I_F \leq 1$) and two-point functions of bosonic fields (for which $I_F \leq 0$) may have a non-negative superficial degree of UV divergence and thus potentially UV diverging. This conclusion is the same as that from the continuum power counting. We then repeat the argument in Introduction based on the gauge invariance and finally we infer that only one-loop scalar two-point functions are potentially logarithmically UV diverging and may suffer from lattice artifacts in the $a \to 0$ limit.[19]

3.4 Global symmetries

Before concluding this section, we briefly comment on other global symmetries besides the supersymmetry in our formulation. The target theory possesses two U(1) symmetries, one is vector-like (2.9) and another is chiral (2.10). With a use of the Wilson-Dirac operator, both symmetries are broken for finite lattice spacings. The fermion number symmetry in 2 dimensions (2.9) was originally the *chiral* $\mathrm{U}(1)_R$ symmetry in the $\mathcal{N} = 1$ SYM theory in 4 dimensions (2.5) and this chiral symmetry is explicitly broken by the Wilson term as usual. The chiral U(1) symmetry (2.10), on the other hand, was a rotation in the 2-3 plane in 4 dimensions and it is broken by a lattice structure, though a discrete subgroup of it is preserved by our construction as eq. (3.9).

With a use of the overlap-Dirac operator, owing to the Ginsparg-Wilson relation $\Gamma D + D\Gamma = aD\Gamma D$ [35], the chiral $\mathrm{U}(1)_R$ transformation in 4 dimensions (2.5) can be modified as [36]

$$\delta_\epsilon \lambda(x) = i\epsilon\Gamma\left(1 - \frac{1}{2}aD\right)\lambda(x)\,, \tag{3.32}$$

so that the lattice action is invariant even with finite lattice spacings. Unfortunately, the fermion integration measure is not invariant under this transformation, producing a non-trivial jacobian

$$\exp\left\{a^2 \sum_{x\in\Gamma} \epsilon \frac{ia}{2}\,\mathrm{tr}\{\Gamma D(x,x)\}\right\}. \tag{3.33}$$

In the continuum limit, this jacobian becomes unity as is consistent with the fact that the transformation (2.5) is vector-like in a 2 dimensional sense (see eq. (2.9)). For finite lattice spacings, however, there is no reason to expect that the jacobian (3.33) is unity. As a result, the fermion number U(1) (2.9) as well as the chiral U(1) (2.10) is not manifest in our formulation with the overlap-Dirac operator.

[19] We note that our power counting in this subsection which concentrates only on UV divergences does not immediately lead to a rigorous proof for our lattice action to possess a supersymmetric continuum limit to all orders of perturbation theory. For such a proof, we have to properly treat an effect of IR divergences associated to massless propagators in a certain way. What we wanted to demonstrate here is that there do not emerge "exotic" lattice artifacts such as an example in Introduction which spoil a conclusion of the continuum power counting.

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Note that, however, after adding the mass counter terms (3.10), all correlation functions of elementary fields will coincide with continuum ones in the continuum limit, as we have argued. Therefore, these U(1) symmetries will be restored in the continuum limit with *either* use of Dirac operators without any further tuning of parameters.[20]

4. Discussion

In this paper, we proposed a lattice formulation of the $\mathcal{N}=(2,2)$ SYM theory in 2 dimensions, which appears to be favored from a view point of numerical simulations. It must be possible to carry out Monte-Carlo simulations with our formulation by present-day (or near-future) available computer resources.[21] In fact, our construction starts with a lattice formulation of the $\mathcal{N}=1$ SYM theory in 4 dimensions [25], which has intensively been studied by numerical simulations [38]. For Monte-Carlo simulations to be executable, we have to add a mass term for the fermion which explicitly breaks the supersymmetry. The Majorana mass term in $d=4$, after the dimensional reduction, becomes to the Majorana mass term for Dirac fermions in $d=2$:[22]

$$\begin{aligned} S_{\mathrm{mass}}[\lambda] &= -a^2 \sum_{x\in\Gamma} im\,\mathrm{tr}\{\lambda(x)C\Gamma\lambda(x)\} \\ &\Rightarrow a^2 \sum_{x\in\Gamma} \left(-m\,\mathrm{tr}\{\psi(x)B\psi(x)\} + m\,\mathrm{tr}\{\overline{\psi}(x)B\overline{\psi}(x)\}\right). \end{aligned} \tag{4.1}$$

(With the overlap-Dirac operator, the fermion determinant is positive definite with this mass term [21].) Thus, we have to take the massless limit $m\to 0$ in addition to the continuum limit $a\to 0$ in simulations.

What kind of observable will be interesting to be explored in numerical simulations? An obvious candidate is mass spectrum of bound states. Mass spectrum and the two-point function of the energy momentum tensor in this $\mathcal{N}=(2,2)$ SYM theory have been investigated [39] by using the supersymmetric discrete light-cone quantization [40]; there, a closing of the mass gap, in accord with an argument [41] on the basis of the 't Hooft anomaly matching condition, is reported. Numerical simulations based on a lattice formulation should be confronted with these results. Before going to study these physical observables, of course, we should be sure about our argument on a restoration of supersymmetry. Thus a restoration of supersymmetric Ward-Takahashi identities has to be firstly confirmed.

Clearly, one is interested in a generalization of our present proposal to other low dimensional extended SYM theories which have been formulated in refs. [1–3]; the $\mathcal{N}=(4,4)$ and $\mathcal{N}=(8,8)$ SYM theories in 2 dimensions and the $\mathcal{N}=2$, $\mathcal{N}=4$ and $\mathcal{N}=8$ SYM theories in 3 dimensions, all these can be obtained by the $\mathcal{N}=1$ SYM theory in higher

[20] In correlation functions with some operators inserted, we have to examine possible breaking of global symmetries caused by corresponding lattice operators. We reserve this issue for future study.

[21] We have determined counter terms which ensure a supersymmetric continuum limit. By a further systematic study along a similar line, it might be possible to carry out the $O(a)$ improvement program [37] which accelerates an approach to a supersymmetric continuum limit.

[22] In this expression, B denotes the "charge conjugation matrix" in $d=2$ such that $B\gamma_\mu B^{-1}=-\gamma_\mu^T$, $B\gamma B^{-1}=-\gamma^T$, $B^{-1}=B^\dagger$ and $B^T=-B$; this is the matrix B_1 of ref. [21].

dimensions via the dimensional reduction. For this issue, it is useful to distinguish two aspects of our present formulation. First, we have defined the $\mathcal{N} = (2,2)$ SYM theory in 2 dimensions by using a dimensional reduction of the $\mathcal{N} = 1$ SYM theory in 4 dimensions. In a similar way, the $\mathcal{N} = 2$ SYM theory in 3 dimensions may be formulated starting with eq. (3.1) just by reducing only $M = 3$ direction, although for this case, we need more general counter terms. (A coefficient of these counter terms is expected to be UV finite, from a power counting and the gauge invariance.) A generalization of this aspect of our formulation to the $\mathcal{N} = 1$ SYM theory in 6 dimensions and in 10 dimensions is however not straightforward. For the $\mathcal{N} = 1$ SYM theory in 6 dimensions, we first have to define a lattice gauge theory in 6 dimensions which contains a single adjoint Weyl fermion. This theory, in a 6 dimensional gauge theoretical sense, is anomalous and we first have to show that possible obstructions in a gauge invariant lattice formulation of chiral gauge theories, such as the one in refs. [30, 42], disappear in a process of the dimensional reduction $6 \to 2$ or $6 \to 3$. (The gauge anomaly in general implies a failure [43] of a lattice formulation along lines of refs. [30, 42].) It is conceivable that this can be shown by imitating an argument of ref. [44]. A generalization to the $\mathcal{N} = 1$ SYM theory in 10 dimensions seems much harder because we do not have a proper local lattice action for the Majorana-Weyl fermion in 10 dimensions for the present [21].

On the other hand, we used the fact that the $\mathcal{N} = (2,2)$ SYM theory in 2 dimensions is super-renormalizable and argued that one-loop calculable mass counter terms ensure a supersymmetric continuum limit. The above SYM theories in 2 and 3 dimensions are all super-renormalizable and thus our argument is equally applied to the above list of theories, although 3 dimensional theories require various type of counter terms besides scalar mass terms. Reality and positivity of the fermion determinant and a compactness of flat directions are of course a separate issue and we have to find some mechanism if these properties desirable for numerical simulations are thought to be kept.

Note added. Quite recently, Elliott and Moore [45] performed independently a similar analysis to ours for the $\mathcal{N} = 2$ Wess-Zumino model and the $\mathcal{N} = 2$ supersymmetric QCD in 3 dimensions.

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References

[1] D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a spatial lattice*, *JHEP* **05** (2003) 037 [hep-lat/0206019];
A.G. Cohen, D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a euclidean spacetime lattice, I. A target theory with four supercharges*, *JHEP* **08** (2003) 024 [hep-lat/0302017]; *Supersymmetry on a euclidean spacetime lattice, II. Target theories with eight supercharges*, *JHEP* **12** (2003) 031 [hep-lat/0307012];

D.B. Kaplan and M. Ünsal, *A euclidean lattice construction of supersymmetric Yang-Mills theories with sixteen supercharges*, *JHEP* **09** (2005) 042 [hep-lat/0503039].

[2] F. Sugino, *A lattice formulation of super Yang-Mills theories with exact supersymmetry*, *JHEP* **01** (2004) 015 [hep-lat/0311021]; *Super Yang-Mills theories on the two-dimensional lattice with exact supersymmetry*, *JHEP* **03** (2004) 067 [hep-lat/0401017]; *Various super Yang-Mills theories with exact supersymmetry on the lattice*, *JHEP* **01** (2005) 016 [hep-lat/0410035].

[3] S. Catterall, *A geometrical approach to $N = 2$ super Yang-Mills theory on the two dimensional lattice*, *JHEP* **11** (2004) 006 [hep-lat/0410052]; *Lattice formulation of $N = 4$ super Yang-Mills theory*, *JHEP* **06** (2005) 027 [hep-lat/0503036].

[4] J. Giedt, *Non-positive fermion determinants in lattice supersymmetry*, *Nucl. Phys.* **B 668** (2003) 138 [hep-lat/0304006]; *The fermion determinant in (4,4) 2D lattice super-Yang-Mills*, *Nucl. Phys.* **B 674** (2003) 259 [hep-lat/0307024].

[5] S. Catterall and S. Karamov, *A lattice study of the two-dimensional Wess Zumino model*, *Phys. Rev.* **D 68** (2003) 014503 [hep-lat/0305002];
S. Catterall and S. Ghadab, *Lattice sigma models with exact supersymmetry*, *JHEP* **05** (2004) 044 [hep-lat/0311042].

[6] J. Giedt, *Deconstruction, 2d lattice super–Yang-Mills and the dynamical lattice spacing*, hep-lat/0405021;
J. Giedt and E. Poppitz, *Lattice supersymmetry, superfields and renormalization*, *JHEP* **09** (2004) 029 [hep-th/0407135];
J. Giedt, R. Koniuk, E. Poppitz and T. Yavin, *Less naive about supersymmetric lattice quantum mechanics*, *JHEP* **12** (2004) 033 [hep-lat/0410041].

[7] M. Ünsal, *Compact gauge fields for supersymmetric lattices*, hep-lat/0504016.

[8] T. Onogi and T. Takimi, *Perturbative study of the supersymmetric lattice theory from matrix model*, hep-lat/0506014.

[9] A. D'Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Twisted superspace on a lattice*, *Nucl. Phys.* **B 707** (2005) 100 [hep-lat/0406029]; *Twisted $N = 2$ exact SUSY on the lattice for BF and Wess-Zumino*, hep-lat/0409092.

[10] S. Catterall and S. Karamov, *Exact lattice supersymmetry: the two-dimensional $N = 2$ Wess-Zumino model*, *Phys. Rev.* **D 65** (2002) 094501 [hep-lat/0108024];
S. Catterall, *Lattice supersymmetry and topological field theory*, *JHEP* **05** (2003) 038 [hep-lat/0301028].

[11] S. Elitzur, E. Rabinovici and A. Schwimmer, *Supersymmetric models on the lattice*, *Phys. Lett.* **B 119** (1982) 165;
S. Cecotti and L. Girardello, *Stochastic processes in lattice (extended) supersymmetry*, *Nucl. Phys.* **B 226** (1983) 417;
N. Sakai and M. Sakamoto, *Lattice supersymmetry and the Nicolai mapping*, *Nucl. Phys.* **B 229** (1983) 173;
Y. Kikukawa and Y. Nakayama, *Nicolai mapping vs. exact chiral symmetry on the lattice*, *Phys. Rev.* **D 66** (2002) 094508 [hep-lat/0207013].

[12] K. Fujikawa, *Supersymmetry on the lattice and the Leibniz rule*, *Nucl. Phys.* **B 636** (2002) 80 [hep-th/0205095]; *$N = 2$ Wess-Zumino model on the $D = 2$ euclidean lattice*, *Phys. Rev.* **D 66** (2002) 074510 [hep-lat/0208015].

[13] M.F.L. Golterman and D.N. Petcher, *A local interactive lattice model with supersymmetry*, *Nucl. Phys.* **B 319** (1989) 307.

[14] T. Reisz, *A power counting theorem for Feynman integrals on the lattice*, *Commun. Math. Phys.* **116** (1988) 81; *A convergence theorem for lattice Feynman integrals with massless propagators*, *Commun. Math. Phys.* **116** (1988) 573;
M. Lüscher, *Selected topics in lattice field theory, Lectures given at Summer School "Fields, Strings and Critical Phenomena"*, Les Houches, France, Jun 28–Aug 5, 1988.

[15] H. Neuberger, *Exactly massless quarks on the lattice*, *Phys. Lett.* **B 417** (1998) 141 [hep-lat/9707022]; *More about exactly massless quarks on the lattice*, *Phys. Lett.* **B 427** (1998) 353 [hep-lat/9801031].

[16] D.B. Kaplan, *A method for simulating chiral fermions on the lattice*, *Phys. Lett.* **B 288** (1992) 342 [hep-lat/9206013];
Y. Shamir, *Chiral fermions from lattice boundaries*, *Nucl. Phys.* **B 406** (1993) 90 [hep-lat/9303005];
V. Furman and Y. Shamir, *Axial symmetries in lattice QCD with Kaplan fermions*, *Nucl. Phys.* **B 439** (1995) 54 [hep-lat/9405004].

[17] H. Neuberger, *Nonperturbative BRS invariance and the Gribov problem*, *Phys. Lett.* **B 183** (1987) 337.

[18] E. Witten, *Constraints on supersymmetry breaking*, *Nucl. Phys.* **B 202** (1982) 253.

[19] L. Brink, J.H. Schwarz and J. Scherk, *Supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 121** (1977) 77.

[20] J. Scherk and J.H. Schwarz, *Dual field theory of quarks and gluons*, *Phys. Lett.* **B 57** (1975) 463.

[21] T. Inagaki and H. Suzuki, *Majorana and Majorana-Weyl fermions in lattice gauge theory*, *JHEP* **07** (2004) 038 [hep-lat/0406026].

[22] H. Nicolai, *A possible constructive approach to (super-ϕ^3)$_4$, 1. Euclidean formulation of the model*, *Nucl. Phys.* **B 140** (1978) 294; *A possible constructive approach to (super-ϕ^3)$_4$, 2. Regularization of the model*, *Nucl. Phys.* **B 156** (1979) 157;
A possible constructive approach to (super-ϕ^3)$_4$, 3. On the normalization of Schwinger functions, *Nucl. Phys.* **B 156** (1979) 177.

[23] P. van Nieuwenhuizen and A. Waldron, *On euclidean spinors and Wick rotations*, *Phys. Lett.* **B 389** (1996) 29 [hep-th/9608174].

[24] N. Maru and J. Nishimura, *Lattice formulation of supersymmetric Yang-Mills theories without fine-tuning*, *Int. J. Mod. Phys.* **A 13** (1998) 2841 [hep-th/9705152].

[25] G. Curci and G. Veneziano, *Supersymmetry and the lattice: a reconciliation?*, *Nucl. Phys.* **B 292** (1987) 555;
Y. Taniguchi, *One loop calculation of SUSY Ward-Takahashi identity on lattice with Wilson fermion*, *Phys. Rev.* **D 63** (2001) 014502 [hep-lat/9906026];
A. Feo, *The supersymmetric Ward-Takahashi identity in 1-loop lattice perturbation theory, I. General procedure*, *Phys. Rev.* **D 70** (2004) 054504 [hep-lat/0305020].

[26] J. Nishimura, *Four-dimensional $N = 1$ supersymmetric Yang-Mills theory on the lattice without fine-tuning*, *Phys. Lett.* **B 406** (1997) 215 [hep-lat/9701013].

[27] H. Kawai, R. Nakayama and K. Seo, *Comparison of the lattice Λ parameter with the continuum lambda parameter in massless QCD*, *Nucl. Phys.* **B 189** (1981) 40.

[28] H. Neuberger, *Vector like gauge theories with almost massless fermions on the lattice*, *Phys. Rev.* **D 57** (1998) 5417 [hep-lat/9710089].

[29] D.B. Kaplan and M. Schmaltz, *Supersymmetric Yang-Mills theories from domain wall fermions*, *Chin. J. Phys.* **38** (2000) 543 [hep-lat/0002030].

[30] M. Lüscher, *Abelian chiral gauge theories on the lattice with exact gauge invariance*, *Nucl. Phys.* **B 549** (1999) 295 [hep-lat/9811032].

[31] H. Fukaya and T. Onogi, *Lattice study of the massive Schwinger model with theta term under Luescher's "admissibility" condition*, *Phys. Rev.* **D 68** (2003) 074503 [hep-lat/0305004]; *θ vacuum effects on the chiral condensation and the η' meson correlators in the two-flavor massive* QED_2 *on the lattice*, *Phys. Rev.* **D 70** (2004) 054508 [hep-lat/0403024].

[32] S. Shcheredin et al., *Testing a topology conserving gauge action in QCD*, hep-lat/0409073; XLF collaboration, W. Bietenholz et al., *Lattice gauge actions for fixed topology*, *AIP Conf. Proc.* **756** (2005) 248 [hep-lat/0412017].

[33] M. Lüscher, *Topology of lattice gauge fields*, *Commun. Math. Phys.* **85** (1982) 39.

[34] P. Hernández, K. Jansen and M. Lüscher, *Locality properties of Neuberger's lattice Dirac operator*, *Nucl. Phys.* **B 552** (1999) 363 [hep-lat/9808010];
H. Neuberger, *Bounds on the Wilson Dirac operator*, *Phys. Rev.* **D 61** (2000) 085015 [hep-lat/9911004].

[35] P.H. Ginsparg and K.G. Wilson, *A remnant of chiral symmetry on the lattice*, *Phys. Rev.* **D 25** (1982) 2649;
P. Hasenfratz, V. Laliena and F. Niedermayer, *The index theorem in QCD with a finite cut-off*, *Phys. Lett.* **B 427** (1998) 125 [hep-lat/9801021];
P. Hasenfratz, *Lattice QCD without tuning, mixing and current renormalization*, *Nucl. Phys.* **B 525** (1998) 401 [hep-lat/9802007].

[36] M. Lüscher, *Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation*, *Phys. Lett.* **B 428** (1998) 342 [hep-lat/9802011].

[37] K. Symanzik, *Continuum limit and improved action in lattice theories, 1. Principles and ϕ^4 theory*, *Nucl. Phys.* **B 226** (1983) 187; *Continuum limit and improved action in lattice theories, 2. $O(n)$ nonlinear sigma model in perturbation theory*, *Nucl. Phys.* **B 226** (1983) 205;
B. Sheikholeslami and R. Wohlert, *Improved continuum limit lattice action for QCD with Wilson fermions*, *Nucl. Phys.* **B 259** (1985) 572;
M. Lüscher, S. Sint, R. Sommer and P. Weisz, *Chiral symmetry and O(a) improvement in lattice QCD*, *Nucl. Phys.* **B 478** (1996) 365 [hep-lat/9605038].

[38] For a review see, I. Montvay, *Supersymmetric Yang-Mills theory on the lattice*, *Int. J. Mod. Phys.* **A 17** (2002) 2377 [hep-lat/0112007];
A. Feo, *Predictions and recent results in SUSY on the lattice*, *Mod. Phys. Lett.* **A 19** (2004) 2387 [hep-lat/0410012].

[39] F. Antonuccio, H.C. Pauli, S. Pinsky and S. Tsujimaru, *DLCQ bound states of $N = (2,2)$ super-Yang-Mills at finite and large-N*, *Phys. Rev.* **D 58** (1998) 125006 [hep-th/9808120];

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M. Harada, J.R. Hiller, S. Pinsky and N. Salwen, *Improved results for $N = (2, 2)$ super Yang-Mills theory using supersymmetric discrete light-cone quantization*, *Phys. Rev.* **D 70** (2004) 045015 [hep-th/0404123].

[40] Y. Matsumura, N. Sakai and T. Sakai, *Mass spectra of supersymmetric Yang-Mills theories in (1+1)-dimensions*, *Phys. Rev.* **D 52** (1995) 2446 [hep-th/9504150].

[41] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [hep-th/9510135].

[42] M. Lüscher, *Weyl fermions on the lattice and the non-abelian gauge anomaly*, *Nucl. Phys.* **B 568** (2000) 162 [hep-lat/9904009].

[43] K. Matsui and H. Suzuki, *Anomalous gauge theories revisited*, *JHEP* **01** (2005) 051 [hep-lat/0412041].

[44] H. Suzuki, *Real representation in chiral gauge theories on the lattice*, *JHEP* **10** (2000) 039 [hep-lat/0009036].

[45] J.W. Elliott and G.D. Moore, *Three dimensional $N = 2$ supersymmetry on the lattice*, hep-lat/0509032.